# MATH 2050C Lecture 19 (Mar 24)

Problem Set 10 posted and due on Apr 1.

**Last time**: $f: A \to \mathbb{R}$ is cts at $c \in A$

iff $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \text{when } x \in A,\ |x - c| < \delta$$

**Case 1**: $c$ is NOT a cluster pt. of $A$

$\Rightarrow$ $f$ is automatically cts at $c$ by def$^{n}$.

**Case 2**: $c$ is a cluster pt. of $A$

$\Rightarrow$ " $f$ is cts at $c$ " $\Leftrightarrow$ " $\lim_{x \to c} f(x) = f(c)$ "

[So, we can use seq. criteria.]

**Q**: How to construct NEW cts functions from OLD ones? (c.f. § 5.2 in textbook)

**Idea**: use "limit theorems".

**Thm 1**: $f, g : A \to \mathbb{R}$ is cts at $c \in A$

$\Rightarrow$ $f \pm g,\ fg,\ \underbrace{f/g}$ is cts at $c \in A$

(**Caution**: require $g(c) \neq 0$)

**Thm 2:** $f: A \to \mathbb{R}$ is cts at $c \in A$

$\Rightarrow$ $\underbrace{\sqrt{f}}_{\text{(require } \geq 0)}$, $|f|$ are cts at $c \in A$

<u>Remark</u>: The proof of these just follows directly from Limit Theorems for limit of functions.

There is a new way to construct functions from old ones that is not available for sequences.

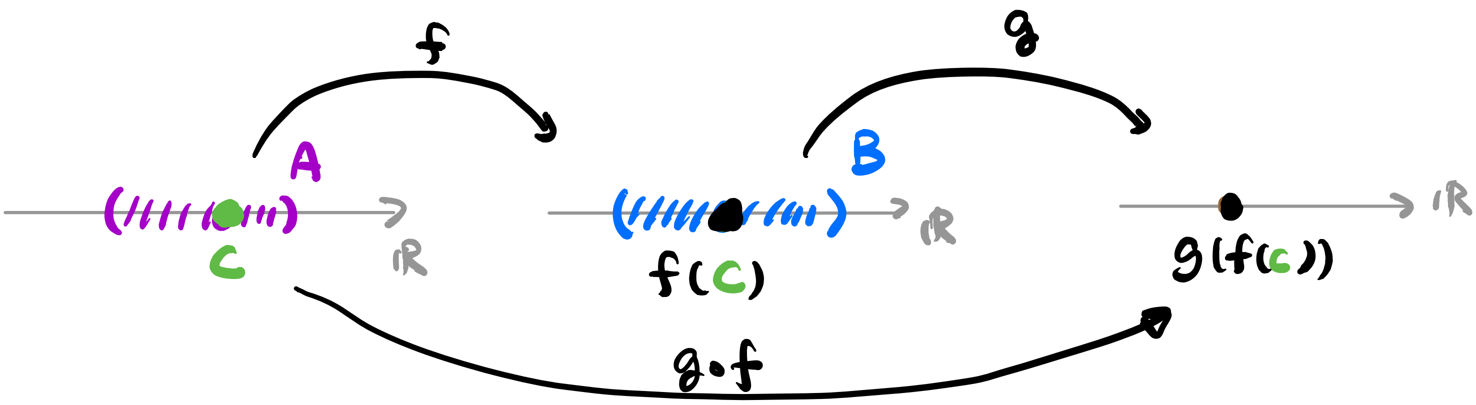
" Composition of functions "

$$\left[\begin{array}{ll} f: A \to \mathbb{R} & \\ & \text{s.t. } f(A) \subseteq B \\ g: B \to \mathbb{R} & \end{array}\right] \quad (*)$$

then we can define the composition

$$g \circ f : A \to \mathbb{R}$$

$$\text{st} \quad (g \circ f)(x) = g(f(x)) \quad \forall x \in A$$



Thm 3 : ( Composition of cts functions)

Assume (*). If $f$ is cts at $c \in A$ and $g$ is cts at $f(c) \in B$, then

$g \circ f$ is cts at $c \in A$

Proof: Either use "$\varepsilon$-$\delta$ def$^{n}$" OR "seg. criteria".

Ex:

Let $\varepsilon > 0$ be fixed but arbitrary.

Denote: $b = f(c) \in B$.

Since $g$ is cts at $b \in B$, by def$^{n}$, for this $\varepsilon > 0$,

$\exists\ \delta_1 = \delta_1(\varepsilon) > 0$ s.t.

$$|g(y) - g(b)| < \varepsilon \quad \text{when } y \in B.\ |y - b| < \delta_1$$

Since $f$ is cts at $c \in A$, for the $\delta_1 > 0$ above,

$\exists\ \delta_2 = \delta_2(\delta_1) > 0$ s.t.

$$|f(x) - f(c)| < \delta_1 \quad \text{when } x \in A.\ |x - c| < \delta_2$$

For this choice of $\delta_2 > 0$, then if $x \in A$ and $|x - c| < \delta_2$, we have

$$|f(x) - f(c)| < \delta_1$$

notice $f(x) \in B$, $b = f(c)$, so

$$|g(f(x)) - g(f(c))| < \varepsilon$$

i.e. $|g \circ f(x) - g \circ f(c)| < \varepsilon$.

∎

Picture: